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Return polynomials for non-intersecting paths above a surface on the directed square lattice

R Brak[1] **and J W Essam**[2]

[1] Department of Mathematics, The University of Melbourne, Parkville, Victoria 3052, Australia
[2] Department of Mathematics, Royal Holloway College, University of London, Egham, Surrey TW20 0EX, UK

E-mail: r.brak@ms.unimelb.edu.au and j.essam@alpha1.rhul.ac.uk

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Abstract

We enumerate sets of n non-intersecting, t-step paths on the directed square lattice which are excluded from the region below the surface $y = 0$ to which they are initially attached. In particular we obtain a product formula for the number of star configurations in which the paths have arbitrary fixed endpoints. We also consider the 'return' polynomial, $\acute{R}_t^{\mathcal{W}}(y;\kappa) = \sum_{m\geqslant 0} \acute{r}_t^{\mathcal{W}}(y;m)\kappa^m$ where $\acute{r}_t^{\mathcal{W}}(y;m)$ is the number of n-path configurations of watermelon type having deviation y for which the path closest to the surface returns to the surface m times. The 'marked return' polynomial is defined by $\acute{u}_t^{\mathcal{W}}(y;\kappa_1) \equiv \acute{R}_t^{\mathcal{W}}(y;\kappa_1+1) = \sum_{m\geqslant 0} \acute{u}_t^{\mathcal{W}}(y;m)\kappa_1^m$ where $\acute{u}_t^{\mathcal{W}}(y;m)$ is the number of marked configurations having at least m returns, just m of which are marked. Both $\acute{r}_t^{\mathcal{W}}(y;m)$ and $\acute{u}_t^{\mathcal{W}}(y;m)$ are expressed in terms of the numbers of paths ignoring returns but introducing a suitably modified endpoint condition. This enables $\acute{u}_t^{\mathcal{W}}(y;m)$ to be written in product form for arbitrary y, but for $\acute{r}_t^{\mathcal{W}}(y;m)$ this can only be done in the case $y = 0$.

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1. Introduction

The problem considered here is of interest in many different contexts, the simplest of which is random walk theory. n random lock-step walkers in one dimension initially occupy the even sites of a semi-infinite lattice and at each tick of a clock move, with equal probability, to one of the adjacent sites subject to the condition that only one walker can be at a given site at any time. Such walkers have been called vicious [1] and are unable to pass one another. In this context our results relate to the probability that the walker closest to the boundary is to be found on the boundary site.

The spacetime trajectories of the random walkers are paths on a fully directed square lattice which is semi-infinite in the space direction. Lattice path problems (as opposed to their continuous analogues) are of considerable significance in enumerative and constructive combinatorics [2, 3]. Here we show that exact recurrence relations and product formulae may be obtained for the number of path configurations of given length and number of surface contacts.

In polymer physics the paths represent networks of non-intersecting directed polymer chains interacting with a surface via contact interactions. The partition function of this system is a sum over network configurations and the Boltzmann weight for a given configuration has a factor κ for each contact the closest chain makes with the surface. This is a non-trivial example of a statistical mechanical system showing an adsorption phase transition which we have found to be exactly solvable for finite size as well as in the thermodynamic limit. The transition takes place at a critical value of $\kappa = \kappa_c = 2$. The scaling behaviour near this transition is expected to be universal and to be found in real polymer networks. Two types of polymer network are normally considered; stars and watermelons [1]. The scaling analysis for watermelons is the subject of a separate paper [4] and here we present only the combinatorial arguments. The partition function in the combinatorial context is known as the return polynomial since for fixed length it is a polynomial in κ and contacts occur when the walk returns to the boundary. For all boundary conditions considered here the first walker starts on the boundary and no κ factor is associated with the initial contact.

For $n = 1$ and $n = 2$ exact expressions for the partition functions of fixed length chains, with various standard end-point conditions, have been found for arbitrary κ [5]. The case when all configurations are given equal weight ($\kappa = 1$), and contacts are not counted is known as the non-interacting case and the bulk case is when no wall is present. The critical exponents describing the asymptotic behaviour of the number of configurations as the chain length approaches infinity have been found in both the non-interacting case [6] and the bulk case [1] for arbitrary n.

In the bulk case the number of star configurations was subsequently expressed exactly for arbitrary fixed length and both free and fixed endpoint conditions in the form of products of ratios of factorials ('product forms') [7, 8]. The formula for fixed endpoints was proved in [8] by evaluation of a Gessel–Viennot determinant [9, 10] and yields as a special case the formula for watermelons with fixed endpoint deviation. The far more difficult proof of the formula for stars with free endpoints was given in [11] and uses a mapping to Young tableaux for which the appropriate sum of Schur functions was known. The numbers of watermelons with free endpoints and $n \leqslant 5$ were proven to satisfy homogeneous linear recurrence relations with polynomial coefficients using Zeilberger's algorithm [12]. For $n = 2$ the relation is of first order yielding a product form but the order increases at odd values of n and the partition functions are generalized hypergeometric functions. For $n = 3$ the partition function is a Heun function.

Product forms have deep combinatorial significance and as such are a major focus of this paper. They can also be readily analysed for their asymptotic behaviour. In the non-interacting surface case our product form (2.7) for fixed endpoint stars is reported in [13] where an alternative proof using knowledge of symplectic characters is given. In the same paper a product formula for stars with free end points is derived by relating it to a problem of enumerating symplectic tableaux previously solved by Proctor [14].

The main result of this paper is a product form for the number of fixed endpoint watermelons having n chains and m contacts (returns). This result is restricted to the case when the endpoint is on the surface. However, for arbitrary deviation a product form is found for the number of configurations with a given number of marked returns. By inclusion and exclusion, the coefficients in the return polynomial are then finite sums of product forms with binomial weights. Alternatively, the numbers of configurations with a given number of marked returns are the coefficients in the partition function when expanded in the variable $\kappa_1 = \kappa - 1$. These results may have significance in symplectic tableaux enumerations.

The calculation is basically the evaluation of the Gessel–Viennot determinant for lattice paths when a weight κ is attached to the surface sites. The elements of this $n \times n$ determinant are single walk partition functions each of which involves κ. It is shown in section 3 that the single walk functions satisfy a recurrence relation which is then used in section 4 to eliminate the dependence on κ from the first $n - 1$ columns of the determinant. Using an expansion formula, derived in section 3, for the elements of the last column yields a determinant which is recognized as a special case of the number of stars with fixed end points and is hence of product form.

The results in this paper follow previous work on single paths on the half plane directed square lattice [5], as well as n non-intersecting paths [15, 16] and their connection with the Bethe ansatz of statistical mechanics [17].

2. Primary definitions and summary of results

The lattice paths are restricted to the upper half plane, $\Xi = \{(t, y) | \, t \in \mathbb{Z}, y \in \mathbb{Z}^+$ and $t + y$ even$\}$, where $\mathbb{Z}$ (resp. $\mathbb{Z}^+$) is the set of integers (resp. non-negative integers). A single such path is a Ballot path defined below.

Definition 1 (Ballot and Dyck paths)**.** *A Ballot path of length t with deviation y is a sequence of vertices* $(v_0, v_1, \ldots, v_t)$, $v_i \in \Xi$, *with* $v_i - v_{i-1} = (1, 1)$ *(an* up *step) or* $(1, -1)$ *(a* down *step),* $i = 1 \ldots t$, $v_0 = (0, 0)$ *and* $v_t = (t, y)$. *Denote the set of Ballot paths of length t, ending at height y by* $\{\wedge\!\wedge\!/\}_{t,y}$. *Ballot paths with zero deviation are known as* Dyck paths *and the set of Dyck paths of length* $2d$ *will be denoted by* $\{\wedge\!\wedge\}_{2d}$.

The number of Ballot paths is well known to be the Ballot number $B_{t,y}$:

$$|\{\wedge\!\wedge\!/\}_{t,y}| \equiv B_{t,y} = \frac{(y+1)t!}{\left(\frac{1}{2}(t+y)+1\right)!\,\left(\frac{1}{2}(t-y)\right)!}. \tag{2.1}$$

These numbers are normally indexed by the numbers of down and up steps $d = \frac{1}{2}(t - y)$ and $e = \frac{1}{2}(t + y)$ so

$$b_{e,d} \equiv B_{e+d,e-d} = \frac{(e-d+1)}{e+1}\binom{e+d}{e}. \tag{2.2}$$

The case $y = 0$ gives the number of Dyck paths

$$|\{\wedge\!\wedge\}_{2d}| \equiv C_d = B_{2d,0} = \frac{1}{d+1}\binom{2d}{d}. \tag{2.3}$$

which is the dth Catalan number.

Definition 2 (Surface, contacts and returns)**.** *The line* $y = 0$ *will be called the* surface. *Any vertex of a Ballot path in common with the surface, is called a* contact. *Contacts other than the initial contact, which is always present, will be known as* returns. *The edge of the path immediately to the left of a return will be known as its* return edge.

In this paper we enumerate the number of configurations of n non-intersecting paths having a given number, m, of returns.

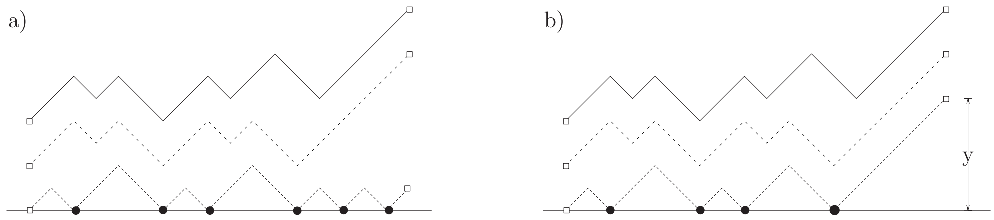

Figure 1. An example of (*a*) a star and (*b*) a watermelon.

Denote the set of Ballot paths of length t with exactly m returns by $\{\wedge\!\wedge\}^m_{t,y}$ and the Ballot path *return polynomial* by

$$\acute{R}^{\mathcal{S}}_t(y,\kappa) \equiv \sum_{m\geqslant 0} \left|\{\wedge\!\wedge\}^m_{t,y}\right| \kappa^m. \tag{2.4}$$

It has been shown that

$$\left|\{\wedge\!\wedge\}^m_{t,y}\right| = B_{t-m-1,\,y+m-1} \tag{2.5}$$

(see for example [5] equation (3.11), replacing m by $m-1$ since in [5] m was the number of contacts). In section 3 this is rederived from a simple bijection which we extend to n paths in section 4.

In the case $n > 1$ we consider two special configurations of non-intersecting paths; 'stars' and 'watermelons' [1].

Definition 3 (Star)**.** *A* star *configuration of length t is a set of n non-intersecting paths (i.e. no pair of paths has any vertices in common) on Ξ which are indexed by $\alpha \in \{1,\ldots,n\}$. The path α begins at $v^i_\alpha = (0, 2(\alpha-1))$ and ends at $v^f_\alpha = (t, y_\alpha)$, see figure 1(a). The number of such configurations whose lowest path makes m returns will be denoted by $\acute{r}^*_t(y_1,y_2,\ldots,y_n;m)$ and the return polynomial is defined as*

$$\acute{R}^*_t(y_1,y_2,\ldots,y_n;\kappa) = \sum_{m\geqslant 0} \acute{r}^*_t(y_1,y_2,\ldots,y_n;m)\kappa^m. \tag{2.6}$$

It is shown in section 4 that the total number of star configurations is given by the product formula

$$\begin{aligned}\acute{R}^*_t(y_1,y_2,\ldots,y_n;1) = &\prod_{1\leqslant\alpha<\beta\leqslant n}\left[\frac{1}{2}(y_\beta-y_\alpha)\left(\frac{1}{2}(y_\alpha+y_\beta)+1\right)\right]\\ &\times\prod_{\alpha=1}^{n}\left[\frac{(t+2\alpha-2)!\,(y_\alpha+1)}{\left(\frac{1}{2}(t+y_\alpha)+n\right)!\,\left(\frac{1}{2}(t-y_\alpha)+n-1)!\right)}\right].\end{aligned} \tag{2.7}$$

Definition 4 (Watermelon)**.** *A* watermelon *configuration of length t and* deviation y *is a star configuration with $y_\alpha = y + 2(\alpha-1)$, see figure 1(b). The number of these watermelon configurations having m returns will be denoted by $\acute{r}^{\mathcal{W}}_t(y;m)$ and the return polynomial is*

$$\acute{R}^{\mathcal{W}}_t(y\,;\kappa) = \sum_{m\geqslant 0}\acute{r}^{\mathcal{W}}_t(y;m)\kappa^m. \tag{2.8}$$

From (2.7) we get the total number of watermelons with fixed deviation y.

$$\acute{R}^{\mathcal{W}}_t(y;1) = \prod_{1\leqslant\alpha<\beta\leqslant n}[(\beta-\alpha)(y+\alpha+\beta-1)]$$

$$\times \prod_{\alpha=1}^{n} \left[\frac{(t+2\alpha-2)!(y+2\alpha-1)}{\left(\frac{1}{2}(t+y)+\alpha+n-1\right)!\left(\frac{1}{2}(t-y)-\alpha+n)!\right)} \right]$$

$$= \prod_{\alpha=1}^{n} \left[\frac{(t+2\alpha-2)!(\alpha-1)!(y+\alpha)_\alpha}{\left(\frac{1}{2}(t+y)+\alpha+n-1\right)!\left(\frac{1}{2}(t-y)+\alpha-1)!\right)} \right]. \tag{2.9}$$

In a parallel work Krattenthaler *et al* [20] derived a product formula for the total numbers of stars which do not go below the surface:

$$\sum_{0\leqslant y_1<y_2<\cdots<y_n} \acute{R}_t^*(y_1, y_2, \ldots, y_n; 1) = \prod_{\alpha=1}^{n} \prod_{\beta=1}^{(t+s)/2} \prod_{\gamma=1}^{(t-s)/2} \frac{\alpha+\beta+\gamma-1}{\alpha+\beta+\gamma-2}$$

$$= \prod_{\alpha=1}^{n} \frac{(\alpha-1)!(t+\alpha-1)!}{\left(\frac{1}{2}(t+s)+\alpha-1\right)!\left(\frac{1}{2}(t-s)+\alpha-1\right)!} \tag{2.10}$$

where $s \equiv t \pmod 2$. They also gave asymptotic forms for this and the number of watermelons with free end condition (i.e. no fixed deviation) for which no product form was found.

Definition 5 (Marked-return stars)**.** *A* marked-return star *is a star with some subset of its returns marked.*

In the case $n = 1$ we denote the set of all Ballot paths of length t ending at height y with exactly m marked returns by $\{$$\}^m_{t,y}$.

Substituting $\kappa = 1 + \kappa_1$ in a star return polynomial and expanding in powers of κ_1 gives 2^m terms for each star configuration having m returns, since a given return may either be associated with a factor 1 or κ_1. There is a clear bijection between the terms having m' factors of κ_1 and stars having a subset of m' returns marked. We therefore call the polynomial in κ_1 a 'marked return polynomial' since it is the generating function for the enumeration of stars with a given number of marked returns. For a single Ballot path the marked return polynomial is

$\acute{U}_t^{\mathcal{S}}(y, \kappa_1) = \sum_{m\geqslant 0} \left|\{$$\}^m_{t,y}\right| \kappa_1^m.$ (2.11)

In the case of watermelons, which are the main subject of this paper,

$$\acute{U}_t^{\mathcal{W}}(y; \kappa_1) \equiv \acute{R}_t^{\mathcal{W}}(y; 1+\kappa_1) = \sum_{m'\geqslant 0} \acute{u}_t^{\mathcal{W}}(y; m')\kappa_1^{m'} \tag{2.12}$$

where $\acute{u}_t^{\mathcal{W}}(y; m')$ is the number of watermelon configurations in which the path nearest to the surface has m' marked returns and is related to $\acute{r}_t^{\mathcal{W}}(y; m)$ by

$$\acute{u}_t^{\mathcal{W}}(y; m') = \sum_{m\geqslant m'} \binom{m}{m'} \acute{r}_t^{\mathcal{W}}(y; m) \tag{2.13}$$

and conversely

$$\acute{r}_t^{\mathcal{W}}(y; m) = \sum_{m'\geqslant m} (-1)^{m'-m} \binom{m'}{m} \acute{u}_t^{\mathcal{W}}(y; m'). \tag{2.14}$$

One reason for introducing the marked return polynomial is that we have found a product form for the coefficient $\acute{u}_t^{\mathcal{W}}(y; m')$ which is given in theorem 5. For the return polynomial we have only found a product form for $\acute{r}_t^{\mathcal{W}}(0; m)$, the number of watermelons attached to the surface at both ends having m returns; see theorem 6. For $y > 0$, $\acute{r}_t^{\mathcal{W}}(0; m)$ factorizes to some extent but there is a residual polynomial in t, y and m of degree $n - 1$ in each of these variables.

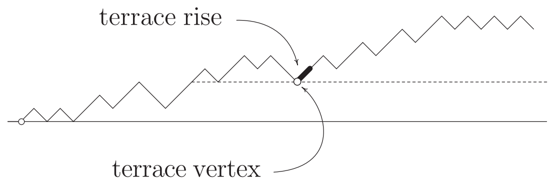

Figure 2. An example of a height 3 terrace showing the terrace rise and terrace vertex.

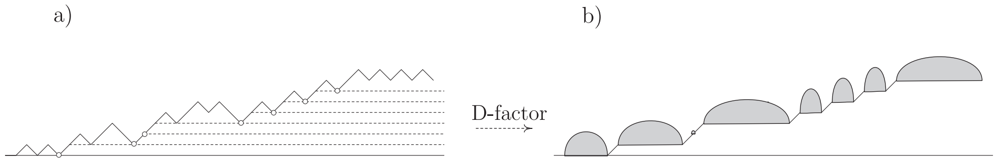

Figure 3. An example (*a*) of the use of terraces to uniquely factorize the Ballot path into *D*-factors (*b*).

A second reason is that in case $n = 1$ it was shown in [5] that the number of Ballot paths with deviation y and m marked returns is given by

$$\left|\{\ \}^{m}_{t,y}\right| = B_{t,y+2m}. \tag{2.15}$$

Here we show that this follows from a simple bijection between Ballot paths with m marked returns and the set of all Ballot paths of the same length which terminate at a distance $2m$ further away from the surface (see section 3). The corresponding bijection in the case of unmarked returns, which gave rise to (2.5), is between paths of different lengths.

3. Combinatorial enumeration of Ballot paths with fixed numbers of returns and marked returns

Definition 6 (Terraces and terrace rises)**.** *A* terrace *of height $h \geqslant 0$ is a horizontal line at height $y = h$. For a Ballot path with deviation $y > h$ we call the rightmost intersection with the terrace, the* terrace vertex *and the up step immediately to the right of the terrace vertex we call the height h the* terrace rise *(see figure 2).*

Definition 7 (Dyck factor)**.** *Consider a Ballot path of length $2d + y$ and height y and draw terraces at heights $0, 1, 2, \ldots, y-1$. The y terrace rises partition the rest of the path into $y+1$ (possibly empty) sub-paths, the first of which is a Dyck path and the other y sub-paths are Dyck paths relative to the terraces, see figure 3. We refer to each Dyck sub-path as a* Dyck factor *or* D-factor.

This factorization into *D*-factors gives the following well-known lemma [18] by summing over the possible lengths of the sub-paths.

Lemma 1. *The number of Ballot paths of length $2d + y$ ending at height y is given in terms of a convolution of $y + 1$ Catalan numbers by*

$$B_{2d+y,y} \equiv \left|\{\ \}_{2d+y,y}\right| = \sum_{d_1\geqslant 0}\sum_{d_2\geqslant 0}\cdots\sideset{}{'}\sum_{d_{y+1}\geqslant 0}\prod_{\alpha=1}^{y+1} C_{d_\alpha} \tag{3.1}$$

where the $'$ on the last sum denotes the restriction $\sum_{\alpha=1}^{y+1} d_\alpha = d$.

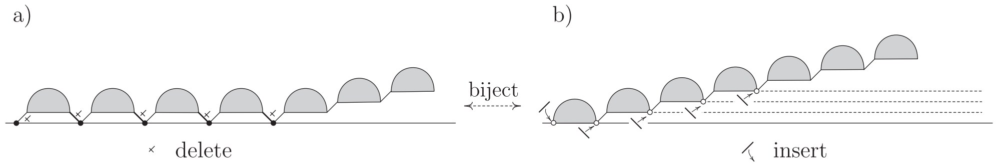

Figure 4. A schematic illustration (*a*) of a Ballot path of length $2d + y$ (with $m = 4$ returns) and height $y = 3$ and its bijection (*b*) to Ballot path of length $2d + y - m - 1$ ending at height $y + m - 1$.

The following enumerations of Ballot paths with fixed numbers of returns and marked returns may therefore also be expressed in terms of convolutions of Catalan numbers.

3.1. The number of Ballot paths with fixed deviation and a given number of returns

A bijection between Ballot paths with exactly m returns and Ballot paths which are $m + 1$ steps shorter but have final deviation increased by $m - 1$ units, illustrated in figure 4, gives the following lemma. A similar construction will be used when we consider the n-path extension later in the paper.

Lemma 2. *The number of Ballot paths of length* $2d + y$ *ending at height* y *with exactly* m *returns is equal to the total number of Ballot paths of length* $2d + y - m - 1$ *and height* $y + m - 1$ *and hence*

$$\left|\{\triangle\!\triangle\!\angle\}^{m}_{2d+y,y}\right| = B_{2d+y-m-1,y+m-1}. \tag{3.2}$$

Proof (Bijection Γ_κ)**.** The lemma is proved by a simple bijection, $\Gamma_\kappa : \mathcal{B} \leftrightarrow \mathcal{B}'$ where $\mathcal{B}$ is the set of Ballot paths of length $2d + y$ and height y with exactly m returns, and $\mathcal{B}'$ is the set of all Ballot paths of length $2d + y - m - 1$ and height $y + m - 1$.

First $\Gamma_\kappa : \mathcal{B} \to \mathcal{B}'$: Given $B \in \mathcal{B}$ contract (i.e. delete the edge and reconnect the path) each of the m return edges, and contract the leftmost edge. Each of the m return edge contractions increases the end height of the path by 1 and moves the end back one unit, whilst the contraction of the first edge shortens the path by 1 further edge and decreases its height by 1, giving a path in $\mathcal{B}'$.

Second $\Gamma_\kappa : \mathcal{B} \leftarrow \mathcal{B}'$: Given $B' \in \mathcal{B}'$ draw m terraces at heights $y = 0, 1, \ldots, m - 1$. Insert a down edge immediately before each of the m terrace rises and insert an up edge at the initial vertex of the path. This results in a Ballot path $m + 1$ steps longer ending $m - 1$ units lower. Each inserted edge (except the first) corresponds to a return edge of the resulting path and no other return edges are created, thus we have a path in $\mathcal{B}$. $\square$

Remark 1. The number of Ballot paths of length $2d + y$ ending at height y with exactly m returns is also in bijection to the number of Ballot paths of length $2d + y - m$ and height $y + m$ with *no* returns. This is observed by modifying figure 4 so that the leftmost edge is not contracted/inserted.

3.2. The number of Ballot paths with fixed deviation and a given number of marked returns

We now give the combinatorial interpretation of (2.15) which is equivalent to our next lemma.

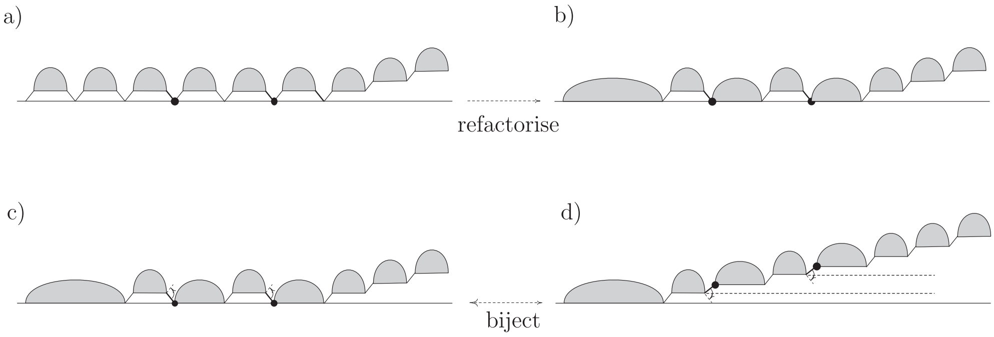

Figure 5. (*a*) A schematic illustration of a Ballot path of length $2d + y$ with $m = 2$ (i.e. two of the six returns marked) and (*b*) a re-factorization showing only the marked return edges which makes it easier to see that the final path (in (*d*)) is a Ballot path. (*c*) The bijection to a Ballot path (*d*) of length $2d + y$ ending at height $y + 2m$.

Lemma 3 $\left(\{\cdots\}^m_{2d+y,y} \overset{\text{biject}}{\longleftrightarrow} \{\cdots\}_{2d+y,y+2m}\right)$**.** *The number of Ballot paths of length* $2d + y$ *with* m *marked returns ending at height* y *is equal to the number of unmarked Ballot paths of the same length ending at height* $y + 2m$:

$$\left|\{\cdots\}^m_{2d+y,y}\right| = B_{2d+y,y+2m}. \tag{3.3}$$

Proof (Bijection Γ_{κ_1})**.** The required bijection, Γ_{κ_1}, is defined as follows:

Rotate each return edge incident on a marked return counterclockwise by 90° (equivalent to replacement by an up step). As illustrated in figure 5, this produces a Ballot path of the same length, but ending at height $y + 2m$. Each rotated return edge becomes a terrace rise since the vertices of the sub-path to the right of the return vertex map to vertices at least as high as the mapped return vertex.

To go from a Ballot path of length $2d + y$ ending at height $y + 2m$ to that ending at height y, draw m terraces at *odd* heights, $y = 1, 3, \ldots, 2m - 1$, then rotate each terrace rise 90° clockwise (equivalent to replacement by a down step). Each rotated terrace rise becomes a return edge of the resulting Ballot path which should be marked to distinguish it from other return edges present or created. □

4. Simplification of the Gessel–Viennot determinant for the watermelon return and marked return polynomials

For $n > 1$ we take as our starting point the following theorem:

Theorem 1. [10]. *Let* $Z(\mathbf{v^i}, \mathbf{v^f})$ *be a weighted sum over configurations of* n *non-intersecting paths, in which path* α *starts at* $v^i_\alpha = \left(t^i_\alpha, y^i_\alpha\right)$ *and ends at* $v^f_\alpha = \left(t^f_\alpha, y^f_\alpha\right)$. *Suppose that the weight attached to a given path is a product of weights associated with vertices and arcs visited by the paths. If there is at least one non-intersecting configuration and all path configurations connecting the initial vertices to any permutation of the terminal vertices* (*other than the identity*) *have at least one intersection then*

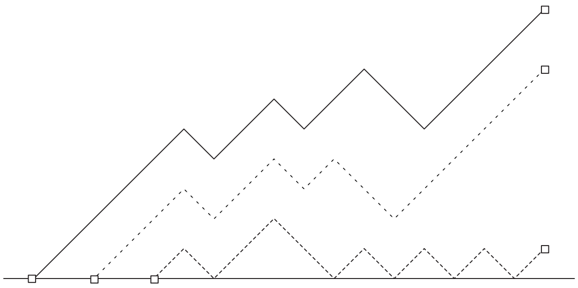

Figure 6. An example of a star with initial vertices extended backwards.

$$Z(\mathbf{v^i}, \mathbf{v^f}) = \begin{vmatrix} Z\left(v_1^i, v_1^f\right) & Z\left(v_1^i, v_2^f\right) & \dots & Z\left(v_1^i, v_n^f\right) \\ Z\left(v_2^i, v_1^f\right) & Z\left(v_2^i, v_2^f\right) & \dots & Z\left(v_2^i, v_n^f\right) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ Z\left(v_n^i, v_1^f\right) & Z\left(v_n^i, v_2^f\right) & \dots & Z\left(v_n^i, v_n^f\right) \end{vmatrix} \tag{4.1}$$

where $Z\big(v_\alpha^i, v_\beta^f\big)$ *is the weighted sum over configurations of a single path starting at vertex* v_α^i *and ending at vertex* v_β^f.

In applying this theorem to the derivation of return (marked return) polynomials for stars (and in particular watermelons) we extend the paths backwards by the least number of steps required to reach the t-axis so that the initial vertex $(0, 2(\alpha-1))$ becomes $v_\alpha^i = (-2(\alpha-1), 0)$ (see figure 6). We call the resulting configurations 'grounded stars'. The operation may be uniquely reversed so that the numbers of stars and grounded stars are equal and in what follows we enumerate grounded stars (watermelons) having m returns (marked returns). The individual paths of a grounded star are Ballot paths for which the return (marked return) polynomials are known from the previous section. In the case of the return (marked return) polynomials, $\acute{R}_t^{\mathcal{W}}(y;\kappa)\big(\acute{U}_t^{\mathcal{W}}(y;\kappa_1)\big)$ is determined by expressing $Z\big(v_\alpha^i, v_\beta^f\big)$ in terms of the single path polynomials $\acute{R}_t^{\mathcal{S}}(y;\kappa)\big(\acute{U}_t^{\mathcal{S}}(y;\kappa_1)\big)$.

For marked return polynomials we use a recurrence relation relating elements in adjacent columns in order to show that it is possible to set $\kappa_1 = 0$ (or $\kappa = 1$) in all but the last column without changing the value of the determinant. This enables the number of watermelons with a given number of marked returns to be expressed in terms of the total number of stars which is given by (2.7).

For return polynomials the situation is more complicated. First, the corresponding recurrence relation relates the polynomials for Ballot paths ending at (t, y) to those for paths ending at $(t-1, y+1)$. To obtain the formula for watermelon polynomials it is therefore necessary to further extend the paths forward by the least number of steps to reach the line $\mathcal{L}$ through $(t, y+2n-2)$ having slope -1 (see figure 7(a)). We call the resulting configurations 'extended grounded watermelons'. Second, the resulting determinant has $\kappa = 0$ in all but the last column and the number of watermelons with a given number of returns is related to the the total number of n-path configurations with no returns that end on $\mathcal{L}$ and are therefore not stars.

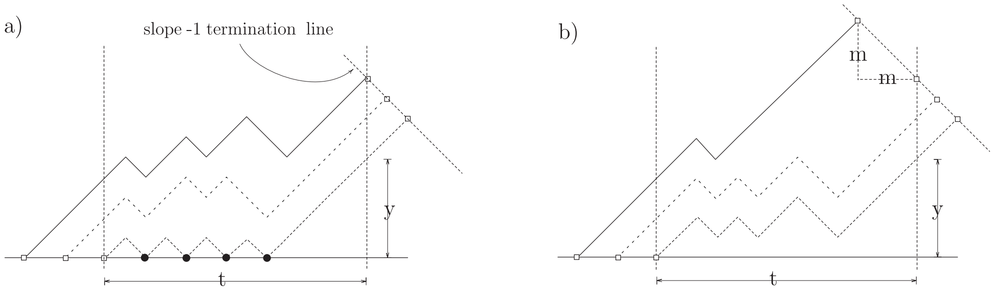

Figure 7. (*a*) An example of a watermelon with $m = 4$ returns (black dots) with the initial vertices extended south-west backwards and final vertices extended in the north-east direction to the slope -1 line. (*b*) A combinatorial proof shows that the configurations in (*a*) are equinumerous with watermelons whose uppermost path ends m higher up the slope -1 line and the lowest path has *no* returns.

In addition to the derivation using recurrence relations, the simplified determinants are also obtained by combinatorial methods. The starting point for these methods is the expansion of the Gessel–Viennot determinant

$$Z(\mathbf{v^i}, \mathbf{v^f}) = \sum_{\sigma \in P_n} \epsilon_\sigma \prod_{\beta=1}^{n} Z\left(v^i_{\sigma_\beta}, v^f_\beta\right) \tag{4.2}$$

where P_n is the set of permutations of $\{1, 2, \ldots, n\}$ and σ_β is the image of β under the permutation $\boldsymbol{\sigma}$. If

$$Z\left(v^i_{\sigma_\beta}, v^f_\beta\right) = \sum_{m_\beta \geqslant 0} z_{m_\beta}\left(v^i_{\sigma_\beta}, v^f_\beta\right) x^{m_\beta} \tag{4.3}$$

then

$$[x^m] Z(\mathbf{v^i}, \mathbf{v^f}) = \sum_{\sigma \in P_n} \epsilon_\sigma \sum_{\mathbf{m}_n \in \mathbb{K}^m_n} \prod_{\beta=1}^{n} z_{m_\beta}\left(v^i_{\sigma_\beta}, v^f_\beta\right) \tag{4.4}$$

where $\mathbb{K}^m_n$ is the set of compositions of m into exactly n parts, i.e. the set of n-tuples, $\mathbf{m}_n = (m_1, m_2, \ldots, m_n)$, $m_\alpha \geqslant 0$, such that $\sum_{\alpha=1}^{n} m_\alpha = m$. Given $k \leqslant n-1$, we may partition $\mathbb{K}^m_n$ as $\mathbb{K}^m_n = K^m_{n,k} \cup \bar{K}^m_{n,k}$ where $K^m_{n,k}$ is the set of compositions in which $m_\beta > 0$ for at least one $\beta \in \{1, 2, \ldots, k\}$. Note, $K^m_{n,k} \cap \bar{K}^m_{n,k} = \phi$. In the next two sections it will be shown that if $Z\left(v^i_\alpha, v^f_\beta\right)$ is a Ballot path return (marked return) polynomial and $x = \kappa$ $(x = \kappa_1)$ then with the above boundary conditions the sum over $\mathbf{m}_n \in K^m_{n,k}$ is zero. This is achieved by constructing a bijection which connects configurations corresponding to permutations of opposite parity resulting in cancellation. Hence, the sum over compositions in (4.4) may be restricted to $\bar{K}^m_{n,k}$ which is equivalent to setting $x = 0$ in the first k columns of the Gessel–Viennot determinant.

4.1. Return polynomial for watermelon configurations attached to a surface with fixed endpoint deviation

The return polynomial for watermelons with fixed deviation y is equal to that for extended grounded watermelons, for which path α starts at $v^i_\alpha = (-2(\alpha-1), 0)$ and ends at $v^f_\alpha = (t + n - \alpha, y + n + \alpha - 2)$ on the above line $\mathcal{L}$. Thus theorem 1 gives

$$\acute{R}^{\mathcal{W}}_t(y, \kappa) = \det(\acute{R}^{\mathcal{S}}_{t+n+2\alpha-\beta-2}(y + n + \beta - 2; \kappa))_{\alpha,\beta=1\cdots n}. \tag{4.5}$$

The following theorem shows that for $1 \leqslant k \leqslant n-1$ we may set $\kappa = 0$ in the first k columns of (4.5) without changing the value of the determinant.

Theorem 2. *For any k such that* $1 \leqslant k \leqslant n-1$ *we have*

$$\acute{R}_t^{\mathcal{S}}(y,\kappa) = \det(M_{\alpha\beta}(k))_{\alpha,\beta=1\ldots n} \tag{4.6}$$

where

$$M_{\alpha\beta}(k) = \begin{cases} \acute{R}^{\mathcal{S}}_{t+n+2\alpha-\beta-2}(y+n+\beta-2,0) & \text{for} \quad \beta \leqslant k \\ \acute{R}^{\mathcal{S}}_{t+n+2\alpha-\beta-2}(y+n+\beta-2,\kappa) & \text{for} \quad \beta > k \end{cases} \tag{4.7}$$

and $y \geqslant 0$. *In particular when* $k = n-1$

$$\acute{R}_t^{\mathcal{W}}(y,\kappa) = \begin{vmatrix} \acute{R}^{\mathcal{S}}_{t+n-1}(y+n-1,0) & \ldots & \acute{R}^{\mathcal{S}}_{t+1}(y+2n-3,0) & \acute{R}^{\mathcal{S}}_{t}(y+2(n-1),\kappa) \\ \acute{R}^{\mathcal{S}}_{t+n+1}(y+n-1,0) & \ldots & \acute{R}^{\mathcal{S}}_{t+3}(y+2n-3,0) & \acute{R}^{\mathcal{S}}_{t+2}(y+2(n-1),\kappa) \\ \vdots & \vdots & \vdots & \vdots \\ \acute{R}^{\mathcal{S}}_{t+3(n-1)}(y+n-1,0) & \ldots & \acute{R}^{\mathcal{S}}_{t+2n-1}(y+2n-3,0) & \acute{R}^{\mathcal{S}}_{t+2(n-1)}(y+2(n-1),\kappa) \end{vmatrix}. \tag{4.8}$$

Algebraic proof. We show that while $k < n-1$, increasing k leaves the determinant unchanged and the result follows by induction since it is true for $k = 0$. Combining (2.4) and (2.5) gives

$$\acute{R}_t^{\mathcal{S}}(y;\kappa) = \sum_{m\geqslant 0} B_{t-m-1,y+m-1}\kappa^m \tag{4.9}$$

separating off the first term gives the recurrence relation

$$\acute{R}_t^{\mathcal{S}}(y;\kappa) = B_{t-1,y-1} + \kappa\acute{R}^{\mathcal{S}}_{t-1}(y+1;\kappa) \tag{4.10}$$

and applying this to column $k+1$ of determinant (4.5)

$$\acute{R}_{t+n+2\alpha-k-3}(y+n+k-1,\kappa) = B_{t+n+2\alpha-k-4,y+n+k-2} + \kappa\acute{R}^{\mathcal{S}}_{t+n+2\alpha-k-4}(y+n+k,\kappa). \tag{4.11}$$

Hence $\det(M(k))$ is the sum of two determinants the first of which is $\det(M(k+1))$ $\big(\text{since } B_{t-1,y-1} = \acute{R}_t^{\mathcal{S}}(y;0)\big)$ and the second evaluates to zero since it has two proportional columns $\big(\acute{R}^{\mathcal{S}}_{t+n+2\alpha-k-4}(y+n+k,\kappa)$ is column $k+2\big)$.

Combinatorial proof. Let $\{\wedge\!\wedge\!\!\wedge\}^m_{v^i_\alpha,v^f_\beta}$ be the set of all paths on Ξ from vertex v^i_α to vertex v^f_β with exactly m returns. Further define the n-path, $\mathbf{A}$ to be the n-tuple of paths $(A_1, A_2, \ldots, A_n)$, $A_i \in \{\wedge\!\wedge\!\!\wedge\}^m_{v^i_\alpha,v^f_\beta}$ and the sets

$$\Omega^{\pm}_k = \left\{(\boldsymbol{\sigma},\mathbf{A})|\boldsymbol{\sigma} \in P^{\pm}_n, \mathbf{m}_n \in K^m_{n,k}, A_\beta \in \{\wedge\!\wedge\!\!\wedge\}^m_{v^i_{\sigma_\beta},v^f_\beta}\right\} \quad \text{and} \quad \Omega_k = \Omega^+_k \cup \Omega^-_k \tag{4.12}$$

where $P^{\pm}_n$ is the set of even/odd permutations of $\{1,2,\ldots,n\}$.

If we use (4.4) with $Z(\mathbf{v^i},\mathbf{v^f}) = \acute{R}_t^{\mathcal{S}}(y,\kappa)$ and $z_{m_\beta}\big(v^i_{\sigma_\beta},v^f_\beta\big) = \big|\{\wedge\!\wedge\!\!\wedge\}^{m_\beta}_{v^i_{\sigma_\beta},v^f_\beta}\big|$ then in the theorem, (4.6) is equivalent to

$$[\kappa^m]\acute{R}_t^{\mathcal{W}}(y,\kappa) = \sum_{\sigma\in P_n}\epsilon_\sigma \sum_{\mathbf{m}_n\in\bar{K}^m_{n,k}} \prod_{\beta=1}^{n} \left|\{\wedge\!\wedge\!\!\wedge\}^{m_\beta}_{v^i_{\sigma_\beta},v^f_\beta}\right| \tag{4.13}$$

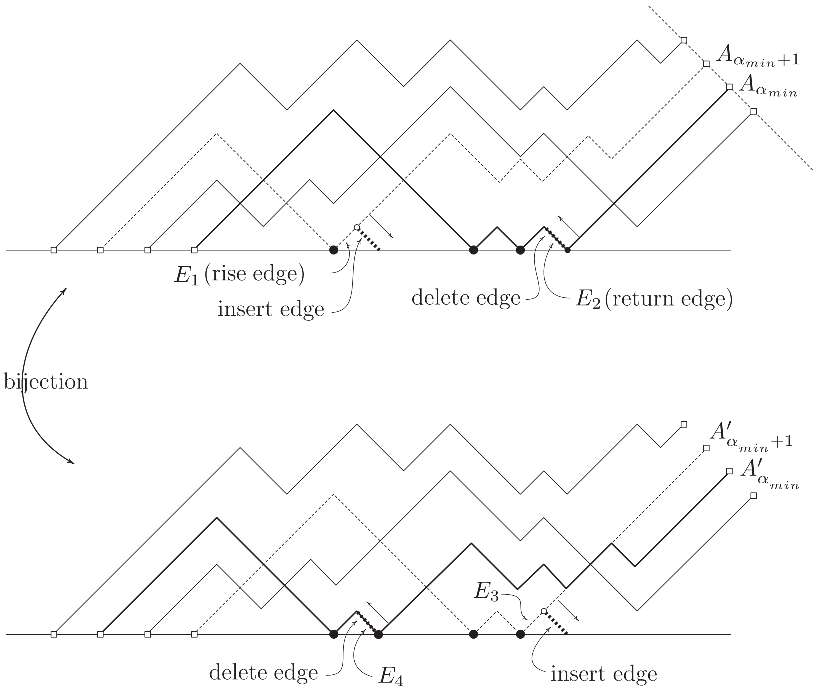

Figure 8. Bijection for n-path return polynomials (see proof of theorem 2).

(since in (4.7) with $\beta \leqslant k$, $[\kappa^{m_\beta}]\acute{R}_t^S(y;\kappa) = 0$ for $m_\beta > 0$). The left-hand side of (4.6) is given by (4.5) and using (4.4) gives the latter as

$$\sum_{\sigma\in P_n}\epsilon_\sigma \sum_{\mathbf{m}_n\in\mathbb{K}_n^m}\prod_{\beta=1}^{n}\left|\{\wedge\wedge\}^{m_\beta}_{v^i_{\sigma_\beta},v^f_\beta}\right| = \sum_{\sigma\in P_n}\epsilon_\sigma \sum_{\mathbf{m}_n\in K_{n,k}^m}\prod_{\beta=1}^{n}\left|\{\wedge\wedge\}^{m_\beta}_{v^i_{\sigma_\beta},v^f_\beta}\right|$$

$$+\sum_{\sigma\in P_n}\epsilon_\sigma \sum_{\mathbf{m}_n\in\bar{K}_{n,k}^m}\prod_{\beta=1}^{n}\left|\{\wedge\wedge\}^{m_\beta}_{v^i_{\sigma_\beta},v^f_\beta}\right|. \tag{4.14}$$

Thus, (4.13) is proven (and hence the theorem), if

$$\sum_{\sigma\in P_n}\epsilon_\sigma \sum_{\mathbf{m}_n\in K_{n,k}^m}\prod_{\beta=1}^{n}\left|\{\wedge\wedge\}^{m_\beta}_{v^i_{\sigma_\beta},v^f_\beta}\right| = 0. \tag{4.15}$$

Using the sets (4.12) shows that the left-hand side of (4.15) is equal to $|\Omega_k^+| - |\Omega_k^-|$. Thus if we show that $|\Omega_k^+| = |\Omega_k^-|$ for $k < n$, then theorem 2 is proven. We do this using a bijection $\Phi_\kappa\colon \Omega_k^+ \leftrightarrow \Omega_k^-$, $k < n$.

Definition of Φ_κ: If $\Omega_k = \phi$, then the result is trivial, thus consider the case $\Omega_k \neq \phi \Rightarrow \exists\, m_\alpha, 1 \leqslant \alpha \leqslant k$ s.t. $m_\alpha > 0 \Rightarrow \exists\, \alpha_{\min} = \min\{\alpha \mid m_\alpha > 0\} \leqslant \kappa < n \Rightarrow \exists$ rightmost return edge, $E_2 \in A_{\alpha_{\min}}$ (see figure 8). Since $\alpha_{\min} < n \Rightarrow \exists\, A_{\alpha_{\min}+1}, v^f_{\alpha_{\min}+1}$ has height coordinate, $y_{\alpha_{\min}+1} \geqslant y+1$ and since $A_{\alpha_{\min}}$ starts on the surface and ends at $y > 0$, $\Rightarrow \exists$ a terrace rise edge $E_1 \in A_{\alpha_{\min}+1}$ of height 0. The pair $(\boldsymbol{\sigma}', \mathbf{A}') = \Phi_\kappa((\boldsymbol{\sigma}, \mathbf{A}))$ is constructed by

- $A'_\alpha = A_\alpha$, for $\alpha \neq \alpha_{\min}, \alpha_{\min}+1$,
- $A'_{\alpha_{\min}}$ is the path $A_{\alpha_{\min}+1}$ but with a down edge E_4 inserted immediately after the edge E_1 (this adds a return to the path),

- $A'_{\alpha_{\min}+1}$ is the path $A_{\alpha_{\min}}$ but with the edge E_2 contracted (this removes a return from the path and creates a terrace rise E_3),
- $\sigma'_\alpha = \sigma_\alpha, \alpha \neq \alpha_{\min}, \alpha_{\min}+1, \sigma'_{\alpha_{\min}} = \sigma_{\alpha_{\min}+1}$ and $\sigma'_{\alpha_{\min}+1} = \sigma_{\alpha_{\min}}$

  (Note, this is a transposition hence $\epsilon_{\sigma'} = -\epsilon_\sigma$)

Since a return is removed from one path ($A_{\alpha_{\min}}$) and added to another ($A_{\alpha_{\min}+1}$) (i.e. $\mathbf{m}'_n = (0, \ldots, 0, m'_{\alpha_{\min}}, m'_{\alpha_{\min}+1}, \ldots)$ with $m'_{\alpha_{\min}} = m_{\alpha_{\min}+1} + 1$ and $m'_{\alpha_{\min}+1} = m_{\alpha_{\min}} - 1$) thus the total number of returns is unchanged. Also, since $A'_{\alpha_{\min}}$ terminates at $v^f_{\alpha_{\min}+1}$, $A'_{\alpha_{\min}+1}$ terminates at $v^f_{\alpha_{\min}}$ and $m'_{\alpha_{\min}} > 0 \Rightarrow (\boldsymbol{\sigma}', \mathbf{A}') \in \Omega_k$. Also, if $(\boldsymbol{\sigma}, \mathbf{A}) \in \Omega^{\pm}_k$ then $(\boldsymbol{\sigma}', \mathbf{A}') \in \Omega^{\mp}_k$ (since $\epsilon_{\sigma'} = -\epsilon_\sigma$).

Is $\Phi^2_\kappa = 1$? Let $(\boldsymbol{\sigma}'', \mathbf{A}'') = \Phi_\kappa((\boldsymbol{\sigma}', \mathbf{A}'))$. Since $m'_\alpha = 0$ for $\alpha < \alpha_{\min}$ and $m'_{\alpha_{\min}} = m_{\alpha_{\min}+1} + 1 > 0 \Rightarrow \alpha'_{\min} = \alpha_{\min}$. The rightmost return edge, E_4 on $A'_{\alpha'_{\min}}$, is immediately after E_1 and hence $A''_{\alpha_{\min}+1} = A_{\alpha_{\min}+1}$. The height 0 terrace rise edge on $A'_{\alpha_{\min}+1}$ is E_3 and hence $A''_{\alpha_{\min}} = A_{\alpha_{\min}}$. Furthermore, since $\alpha'_{\min} = \alpha_{\min} \Rightarrow \boldsymbol{\sigma}'' = \boldsymbol{\sigma}$, thus $(\boldsymbol{\sigma}'', \mathbf{A}'') = (\boldsymbol{\sigma}, \mathbf{A})$ and hence $\Phi^2_\kappa = 1$. Thus Φ_κ is a bijection and hence $|\Omega^+_k| = |\Omega^-_k|$, $k < n$. □

The case $k = n - 1$ has an alternative combinatorial interpretation which arises by using the bijection of lemma 2. The coefficient, $[\kappa^m]\acute{R}^{\mathcal{W}}_t(y, \kappa)$ may be found using the theorem by expanding the elements of the last column of (4.8) using lemma 2 and (4.9). Thus the element on row α becomes $[\kappa^m]\acute{R}^{\mathcal{S}}_{t+2(\alpha-1)}(y + 2(n-1), \kappa) = \acute{R}^{\mathcal{S}}_{t+2(\alpha-1)-m}(y + 2(n-1) + m, 0)$. Using the Gessel–Viennot theorem on the resulting determinant, and since (4.6) is independent of k, gives the following corollary.

Corollary 1. *The non-intersecting n-paths starting at* $\mathbf{v}^i = (v^i_1, \ldots, v^i_n)$ *with* $v^i_\alpha = (-2(\alpha-1), 0)$ *and ending at* $\mathbf{v}^f = (v^f_1, v^f_2, \ldots, v^f_n)$ *with* $v^f_\alpha = (t + n - \alpha, y + n + \alpha - 2)$ (*i.e. extended grounded watermelons*) *where the path adjacent to the surface has exactly m returns, are equinumerous with the non-intersecting n-paths with* no *returns, starting at* $\mathbf{v}^i$ *and ending at* $(v^f_1, \ldots, v^f_{n-1}, v'^f_n)$, *where* $v'^f_n = (t - m - 1, y + 2(n-1) + m - 1)$. *Both are equal in number to watermelons having deviation y and m returns (see figure 9).*

It would be interesting to find a bijection between these two sets.

4.2. Marked-return polynomial for watermelon configurations attached to a surface with fixed endpoint deviation

The marked return polynomial for watermelons with fixed deviation y is equal to that for grounded watermelons for which path α starts at $v^i_\alpha = (-2(\alpha-1), 0)$ and ends at $v^f_\alpha = (t, y + 2(\alpha-1))$. Thus theorem 1 gives

$$\acute{U}^{\mathcal{W}}_t(y, \kappa_1) = \det\left(\acute{U}^{\mathcal{S}}_{t+2\alpha-2}(y + 2\beta - 2; \kappa_1)\right)_{\alpha,\beta=1\cdots n} \tag{4.16}$$

The following theorem shows that for $1 \leqslant k \leqslant n-1$ we may set $\kappa_1 = 0$ in the first k columns of (4.16) without changing the value of the determinant.

Theorem 3. *For any k such that* $1 \leqslant k \leqslant n-1$ *we have*

$$\acute{U}^{\mathcal{W}}_t(y, \kappa_1) = \det(M(k)) \tag{4.17}$$

where $M(k)$ *is the matrix whose* $\alpha - \beta$*th element is*

$$M_{\alpha\beta}(k) = \begin{cases} \acute{U}^{\mathcal{S}}_{t+2(\alpha-1)}(y + 2(\beta-1), 0) & \textit{for} \quad \beta \leqslant k \\ \acute{U}^{\mathcal{S}}_{t+2(\alpha-1)}(y + 2(\beta-1), \kappa_1) & \textit{for} \quad \beta > k \end{cases} \tag{4.18}$$

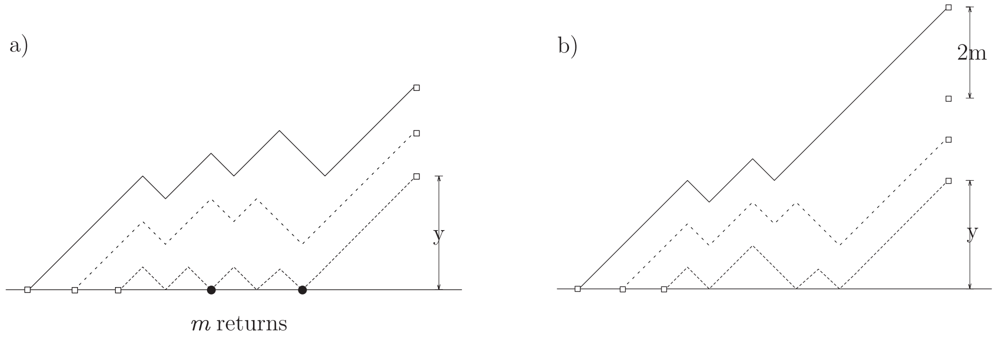

Figure 9. (*a*) An example of a watermelon with *m* marked returns (solid circles) with the initial vertices extended backwards. (*b*) A combinatorial proof shows that the configurations in (*a*) are equinumerous with watermelons with no marked returns and whose uppermost paths end 2*m* higher than in (*a*).

and $y \geqslant 0$. *In particular when* $k = n - 1$

$$\acute{U}_t^{\mathcal{W}}(y, \kappa_1) = \begin{vmatrix} \acute{U}_t^{\mathcal{S}}(y,0) & \dots & \acute{U}_t^{\mathcal{S}}(y+2(n-2),0) & \acute{U}_t^{\mathcal{S}}(y+2(n-1),\kappa_1) \\ \acute{U}_{t+2}^{\mathcal{S}}(y,0) & \dots & \acute{U}_{t+2}^{\mathcal{S}}(y+2(n-2),0) & \acute{U}_{t+2}^{\mathcal{S}}(y+2(n-1),\kappa_1) \\ \acute{U}_{t+4}^{\mathcal{S}}(y,0) & \dots & \acute{U}_{t+4}^{\mathcal{S}}(y+2(n-2),0) & \acute{U}_{t+4}^{\mathcal{S}}(y+2(n-1),\kappa_1) \\ \vdots & \vdots & \vdots & \vdots \\ \acute{U}_{t+2(n-1)}^{\mathcal{S}}(y,0) & \dots & \acute{U}_{t+2(n-2)}^{\mathcal{S}}(y+2(n-2),0) & \acute{U}_{t+2(n-1)}^{\mathcal{S}}(y+2(n-1),\kappa_1) \end{vmatrix} \tag{4.19}$$

Algebraic proof. We show that while $k < n - 1$, increasing k leaves the determinant unchanged and the result follows by induction since it is true for $k = 0$.

Combining (2.11) and (2.15)

$$\acute{U}_t^{\mathcal{S}}(y, \kappa_1) = \sum_{m \geqslant 0} B_{t,y+2m}\, \kappa_1^m \tag{4.20}$$

from which we obtain the recurrence relation

$$\acute{U}_t^{\mathcal{S}}(y; \kappa_1) = B_{t,y} + \kappa_1 \acute{U}_t^{\mathcal{S}}(y+2; \kappa_1) \tag{4.21}$$

and applying this to column $k + 1$ of (4.18)

$$\acute{U}_{t+2(\alpha-1)}^{\mathcal{S}}(y + 2k, \kappa_1) = B_{t+2(\alpha-1),y+2k} + \kappa_1 \acute{U}_{t+2(\alpha-1)}^{\mathcal{S}}(y + 2(k+1), \kappa_1). \tag{4.22}$$

Hence det $(M(k))$ is the sum of two determinants. The first is det $(M(k+1)$ since $B_{t+2(\alpha-1),y+2k} = \acute{U}_{t+2(\alpha-1)}^{\mathcal{S}}(y + 2k, 0)$ and the second evaluates to zero since $\acute{U}_{t+2(\alpha-1)}^{\mathcal{S}}(y + 2(k+1), \kappa_1)$ is column $k + 2$.

Combinatorial proof. This proof follows closely that of theorem 2. Apart from the bijection only the following changes are required. v_β^f becomes $(t, y + 2\beta)$, R becomes U and 'return' becomes 'marked return'. The required bijection is defined as follows.

Definition of Φ_{κ_1}: If $\Omega_k = \phi$, then the theorem follows directly, thus consider the case $\Omega_k \neq \phi \Rightarrow \exists\, m_\alpha, 1 \leqslant \alpha \leqslant k$ s.t. $m_\alpha > 0 \Rightarrow \exists\, \alpha_{\min} = \min\{\alpha \mid m_\alpha > 0\} \leqslant k < n \Rightarrow \exists$ rightmost return edge, $E_2 \in A_{\alpha_{\min}}$ (see figure 10). Since $\alpha_{\min} < n \Rightarrow \exists\, A_{\alpha_{\min}+1}$ and $v_{\alpha_{\min}+1}^f = (t, y_\alpha)$ with $y_\alpha \geqslant y + 2 \Rightarrow \exists$ a terrace rise $E_1 \in A_{\alpha_{\min}+1}$ of height 1. The pair $(\boldsymbol{\sigma}', \mathbf{A}') = \Phi_{\kappa_1}((\boldsymbol{\sigma}, \mathbf{A}))$ is constructed by

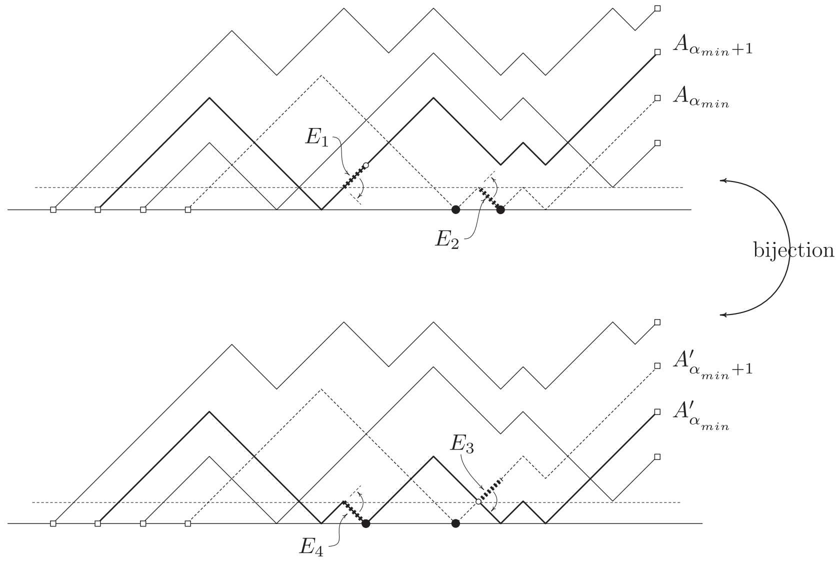

Figure 10. Bijection for κ_1 coefficients.

- $A'_\alpha = A_\alpha$, for $\alpha \neq \alpha_{\min}, \alpha_{\min}+1$,
- $A'_{\alpha_{\min}}$ is the path $A_{\alpha_{\min}+1}$ but with the edge E_1 replaced by a down edge E_4, and a mark is inserted on the right adjacent vertex of E_4,
- $A'_{\alpha_{\min}+1}$ is the path $A_{\alpha_{\min}}$ but with the edge E_2 replaced by an up edge E_3, and the rightmost mark on $A_{\alpha_{\min}}$ removed,
- $\sigma'_\alpha = \sigma_\alpha, \alpha \neq \alpha_{\min}, \alpha_{\min}+1$, $\sigma'_{\alpha_{\min}} = \sigma_{\alpha_{\min}+1}$ and $\sigma'_{\alpha_{\min}+1} = \sigma_{\alpha_{\min}}$

  (Note, this is a transposition hence $\epsilon_{\boldsymbol{\sigma}'} = -\epsilon_{\boldsymbol{\sigma}}$.)

Since a mark is removed from one path and added to another (i.e. $\mathbf{m}'_n = (0, \ldots, 0, m'_{\alpha_{\min}}, m'_{\alpha_{\min}+1}, \ldots)$ with $m'_{\alpha_{\min}} = m_{\alpha_{\min}+1}+1$ and $m'_{\alpha_{\min}+1} = m_{\alpha_{\min}}-1$) the total number of marks is unchanged. Also, since $A'_{\alpha_{\min}}$ terminates at $v^f_{\alpha_{\min}+1}$, $A'_{\alpha_{\min}+1}$ terminates at $v^f_{\alpha_{\min}}$ and $m'_{\alpha_{\min}} > 0$, $\Rightarrow (\boldsymbol{\sigma}', \mathbf{A}') \in \Omega_k$. Since $\epsilon_{\boldsymbol{\sigma}'} = -\epsilon_{\boldsymbol{\sigma}} \Rightarrow$, if $(\boldsymbol{\sigma}, \mathbf{A}) \in \Omega^{\pm}_k$, then $(\boldsymbol{\sigma}', \mathbf{A}') \in \Omega^{\mp}_k$.

Is $\Phi^2_{\kappa_1} = 1$? Let $(\boldsymbol{\sigma}'', \mathbf{A}'') = \Phi_{\kappa_1}((\boldsymbol{\sigma}', \mathbf{A}'))$. Since $m'_\alpha = 0$ for $\alpha < \alpha_{\min}$ and $m'_{\alpha_{\min}} = m_{\alpha_{\min}+1} + 1 > 0 \Rightarrow \alpha'_{\min} = \alpha_{\min}$. The rightmost mark on $A'_{\alpha'_{\min}}$ is adjacent to E_4 and hence $A''_{\alpha_{\min}+1} = A_{\alpha_{\min}+1}$. The height 1 terrace rise edge on $A'_{\alpha_{\min}+1}$ is E_3 and hence $A''_{\alpha_{\min}} = A_{\alpha_{\min}}$. Furthermore, since $\alpha'_{\min} = \alpha_{\min} \Rightarrow \boldsymbol{\sigma}'' = \boldsymbol{\sigma}$, thus $(\boldsymbol{\sigma}'', \mathbf{A}'') = (\boldsymbol{\sigma}, \mathbf{A})$ and hence $\Phi^2_{\kappa_1} = 1$ $\square$

$\left[\kappa_1^m\right]\hat{U}^{\mathcal{W}}_t(y, \kappa_1)$ may be found using the theorem by expanding the elements of the last column of (4.19) using lemma 3 and (4.20). Thus the element on row α becomes $\left[\kappa_1^m\right]\hat{U}^{\mathcal{S}}_{t+2(\alpha-1)}(y+2(n-1), \kappa_1) = \hat{U}^{\mathcal{S}}_{t+2(\alpha-1)}(y+2(n-1)+2m, 0)$. Using the Gessel–Viennot theorem on the resulting determinant gives the following corollary.

Corollary 2. *The non-intersecting n-paths starting at* $\mathbf{v}^i = \left(v^i_1, \ldots, v^i_n\right)$ *with* $v^i_\alpha = (-2(\alpha-1), 0)$ *and ending at* $\left(v^f_1, v^f_2, \ldots, v^f_n\right)$ *with* $v^f_\alpha = (t, y+2(\alpha-1))$ *(i.e. grounded watermelons), where the path adjacent to the surface has m marked returns, are equinumerous with the the non-intersecting n-paths with no marked returns, starting at* $\mathbf{v}^i$ *and ending at* $\left(v^f_1, \ldots, v^f_{n-1}, v'^f_n\right)$*, where* $v'^f_n = (t, y+2(n-1)+2m)$. *Both are equal in number to watermelons with deviation y and m marked returns.*

It would be interesting to find a bijection between these two sets.

4.3. Recurrence relations

In order to make the n dependence explicit let $\acute{R}_t^{(n)}(y, \kappa_1) \equiv \acute{R}_t^{\mathcal{W}}(y, \kappa_1)$. Expanding the determinant (4.8) using Dodgson's condensation formula [19] gives the recurrence relation

$$\begin{aligned}\acute{R}_{t+3}^{(n-2)}(y+3; 0)\acute{R}_t^{(n)}(y; \kappa) = {} & \acute{R}_{t+1}^{(n-1)}(y+1; 0)\acute{R}_{t+2}^{(n-1)}(y+2; \kappa) \\ & - \acute{R}_{t+3}^{(n-1)}(y+1; 0)\acute{R}_t^{(n-1)}(y+2; \kappa)\end{aligned} \tag{4.23}$$

and similarly expanding (4.19)

$$\acute{U}_{t+2}^{(n-2)}(y+2; 0)\acute{U}_t^{(n)}(y; \kappa_1) = \acute{U}_t^{(n-1)}(y; 0)\acute{U}_{t+2}^{(n-1)}(y+2; \kappa_1) - \acute{U}_{t+2}^{(n-1)}(y; 0)\acute{U}_t^{(n-1)}(y+2; \kappa_1). \tag{4.24}$$

The following further recurrence relation will be required in proving theorem 6 which gives the product form for watermelons with zero deviation:

$$\acute{R}_{2d+4}^{(n-2)}(0; 1)\acute{R}_{2d}^{(n)}(0; \kappa) = \kappa^{-2}\left[\acute{R}_{2d}^{(n-1)}(0; 1)\acute{R}_{2d+4}^{(n-1)}(0; \kappa) - \acute{R}_{2d+2}^{(n-1)}(0; 1)\acute{R}_{2d+2}^{(n-1)}(0; \kappa)\right]. \tag{4.25}$$

Proof. Watermelons with zero deviation can be grounded at both ends by extending each path in both the backward and forward directions by the least number of steps required to reach the t-axis. Again this operation is reversible and so we may count doubly grounded watermelons instead of watermelons. The paths which occur in the Gessel–Viennot theorem are then Dyck paths and path α begins at $(-2(\alpha - 1), 0)$ and ends at $(t + 2(\alpha - 1), 0)$. Also substituting the appropriate Dyck path return polynomials into theorem 1 will give the watermelon return polynomial multiplied by a factor κ^{n-1} arising from the additional returns introduced by the terminal grounding. Thus

$$\acute{R}_{2d}^{(n)}(0, \kappa) = \kappa^{-(n-1)} \det\left(\acute{R}_{2(d+\alpha+\beta-2)}^{\mathcal{S}}(0; \kappa)\right)_{\alpha,\beta=1\cdots n} \tag{4.26}$$

The proof now follows closely that of theorem 2 and we give only a brief discussion. Using the recurrence relation

$$\kappa^2 \acute{R}_{2d}^{\mathcal{S}}(0; \kappa) = \kappa^2 \acute{R}_{2d}^{\mathcal{S}}(0; 1) + (\kappa - 1)\acute{R}_{2d+2}^{\mathcal{S}}(0; \kappa) \tag{4.27}$$

which was given in [5] equation (3.29), shows that we may set $\kappa = 1$ in the first $n - 1$ columns of the determinant (4.26) without changing its value. The recurrence relation follows by applying Dodgson's formula [19] to the resulting determinant. $\square$

5. Factorization of determinants and product forms

5.1. Product form for the number of stars attached to a surface with fixed endpoint deviations

Applying theorem 1 to the calculation of the number of grounded stars gives the following formula for the return polynomial of stars of length t where path α has fixed deviation y_α.

$$\acute{R}_t^*(y_1, y_2, \ldots, y_n; \kappa) = \begin{vmatrix} \acute{R}_t^{\mathcal{S}}(y_1; \kappa) & \acute{R}_t^{\mathcal{S}}(y_2; \kappa) & \ldots & \acute{R}_t^{\mathcal{S}}(y_n; \kappa) \\ \acute{R}_{t+2}^{\mathcal{S}}(y_1; \kappa) & \acute{R}_{t+2}^{\mathcal{S}}(y_2; \kappa) & \ldots & \acute{R}_{t+2}^{\mathcal{S}}(y_n; \kappa) \\ \acute{R}_{t+4}^{\mathcal{S}}(y_1; \kappa) & \acute{R}_{t+4}^{\mathcal{S}}(y_2; \kappa) & \ldots & \acute{R}_{t+4}^{\mathcal{S}}(y_n; \kappa) \\ \ldots & \ldots & \ldots & \ldots \\ \acute{R}_{t+2(n-1)}^{\mathcal{S}}(y_1; \kappa) & \acute{R}_{t+2(n-1)}^{\mathcal{S}}(y_2; \kappa) & \ldots & \acute{R}_{t+2(n-1)}^{\mathcal{S}}(y_n; \kappa) \end{vmatrix} \tag{5.1}$$

where $\acute{R}_t^{\mathcal{S}}(y; \kappa) \equiv P\{$$\}_{t,y}(\kappa)$ is the Ballot path return polynomial for a single path.

We evaluate the determinant for $\kappa = 1$ which gives the total number of star configurations.

Theorem 4. *The number of non-intersecting star configurations of n paths in which path α starts at $(0, 2(\alpha-1))$ and ends at (t, y_α) steps without crossing the wall $y = 0$ is given by the product formula*

$$\acute{R}_t^*(y_1, y_2, \ldots, y_n; 1) = \prod_{1\leqslant\alpha<\beta\leqslant n}\left[\frac{1}{2}\left(y_\beta - y_\alpha\right)\left(\frac{1}{2}\left(y_\alpha + y_\beta\right) + 1\right)\right]$$
$$\times \prod_{\alpha=1}^{n}\left[\frac{(t+2\alpha-2)!(y_\alpha+1)}{\left(\frac{1}{2}(t+y_\alpha)+n\right)!\left(\frac{1}{2}(t-y_\alpha)+n-1\right)!)}\right]. \tag{5.2}$$

Proof. With $\kappa = 1$ and $d_\beta = \frac{1}{2}(t - y_\beta)$, the $\alpha - \beta$ element of the determinant is the Ballot number

$$B_{t+2\alpha-2, y_\beta} = \frac{(y_\beta+1)(t+2\alpha-2)!}{(d_\beta+\alpha-1)!(t-d_\beta+\alpha)!} \tag{5.3}$$

and removing common factors from (5.1) gives

$$\acute{R}_t^*(y_1, y_2, \ldots, y_n; 1) = D_n \prod_{\alpha=1}^{n}\frac{(t+2\alpha-2)!(y_\alpha+1)}{(d_\alpha+n-1)!(t-d_\alpha+n)!} \tag{5.4}$$

where D_n is the determinant of the n by n matrix having $\alpha - \beta$ element $(d_\beta+\alpha)_{n-\alpha}(t-d_\beta+\alpha+1)_{n-\alpha}$. The determinant is a polynomial of degree $\frac{1}{2}n(n-1)$ in t and degree $n-1$ in d_β which for $\alpha \neq \beta$ vanishes when $d_\beta = d_\alpha$ or $d_\beta = t - d_\alpha + 1$ and hence by comparing degrees

$$D_n = \prod_{1\leqslant\alpha<\beta\leqslant n}(d_\alpha - d_\beta)(t - d_\alpha - d_\beta + 1). \tag{5.5}$$

□

5.2. Product form for the number of watermelons with exactly m marked returns

Using corollary 2 the number of watermelon configurations with deviation y and m marked returns is given by

$$\acute{u}_t^{\mathcal{W}}(y, m) = \acute{R}_t^*(y, y+2, \ldots, y+2(n-2), y+2(n-1)+2m; 1) \tag{5.6}$$

which together with theorem 4 gives the following product formula.

Theorem 5. *The number of non-intersecting n-path configurations starting at $\mathbf{v}^i = \left(v_1^i, \ldots, v_n^i\right)$ with $v_\alpha^i = (-2(\alpha-1), 0)$ and ending at $\left(v_1^f, v_2^f, \ldots, v_n^f\right)$ with $v_\alpha^f = (t, y+2(\alpha-1))$ (i.e. grounded watermelons), where the path adjacent to the surface has exactly m marked returns, is given by*

$$\acute{u}_t^{\mathcal{W}}(y, m) = \acute{R}_t^{\mathcal{W}}(y; 1) f_t^{(n)}(y, m) \tag{5.7}$$

where

$$f_t^{(n)}(y, m) = \binom{n+m-1}{m}\frac{(y+2n+2m-1)(y+n+m)_{n-1}\left(\frac{1}{2}(t-y)-m+1\right)_m}{(y+n)_n\left(\frac{1}{2}(t+y)+2n\right)_m} \tag{5.8}$$

and $\acute{R}_t^{\mathcal{W}}(y; 1)$ is the product form for the total number of watermelons with deviation y given by (2.9).

5.3. The number of watermelons with exactly m returns

The following result follows from corollary 1 or by substitution of (4.9) into (4.8).

Corollary 3. *The number of n-path watermelon configurations with exactly m returns and fixed deviation y is*

$$\acute{r}_t^{\mathcal{W}}(y,m) = \det(B_{t-m'+n+2\alpha-\beta-3,\,y+m'+n+\beta-3})_{\alpha,\beta=1\cdots n} \tag{5.9}$$

with

$$m' = \begin{cases} 0 & \text{for } \beta < n \\ m & \text{for } \beta = n. \end{cases} \tag{5.10}$$

Now

$$B_{t-m'+n+2\alpha-\beta-3,\,y+m'+n+\beta-3} = \frac{(y+m'+n+\beta-2)(t-m'+n+2\alpha-\beta-3)!}{(e+n+\alpha-2)!(d-m'+\alpha-\beta)!}. \tag{5.11}$$

Removing common factors from rows and columns reduces the determinant (5.9) to one with polynomial elements

$$\acute{r}_t^{\mathcal{W}}(y,m) = D_t^{(n)}(y,m)\frac{(y+m+2n-2)(t-m-1)!}{(e+2n-2)!(d-m)!}\prod_{\alpha=1}^{n-1}\frac{(y+n+\alpha-2)(t+2\alpha-2)!}{(d+\alpha)!(e+n+\alpha-2)!} \tag{5.12}$$

where $D_t^{(n)}(y,m) = \det A$ with

$$A_{i,j} = \begin{cases} (d+i-j+1)_{j-1}(t+2i-1)_{n-j-1} & i,j<n \\ (d+1)_{i-1}(d-m-n+i+1)_{n-i}(t-m)_{2i-2}(t+2i-1)_{2n-2i-1} & i<n,\ j=n \\ (d+n-j+1)_{j-1}(t+2n-2)_{n-j} & i=n,\ j<n \\ (d+1)_{n-1}(t-m)_{2n-2} & i=n,\ j=n. \end{cases} \tag{5.13}$$

When $n=1$, $\det A = 1$ in agreement with lemma 2. Evaluating the determinant for $n=2$ and $n=3$ gives

$$D_t^{(2)}(y,m) = ((d+1)m + y(t+1))(m+1) \tag{5.14}$$

and

$$D_t^{(3)}(x,m) = ((d+1)(d+2)m(m+3) + 2xm(d+1)(t+3) + x(x+2)(t+1)(t+3)) \times (y+1)(m+1)(m+2). \tag{5.15}$$

On further evaluation for increasing values of n it becomes clear that $D_t^{(2)}(y,m)$ is the product of the following simple factors and an 'ugly' polynomial [20] in the variables t, y and m of degree $n-1$ in each of the variables:
for n even

$$(m+1)_{n-1}\prod_{i=1}^{n/2-1}(y+2i-1)_{n-1} \tag{5.16}$$

and for n odd

$$(m+1)_{n-1}\prod_{i=1}^{(n-1)/2}(y+2i-1)_n. \tag{5.17}$$

When $y=0$ the ugly polynomial factorizes to give the following theorem.

Theorem 6. *The number of watermelons of length 2d attached to the surface at both ends (i.e. with zero deviation) and having exactly m returns is given by*

$$\acute{r}^{(n)}_{2d}(0; m) \equiv \acute{r}^{\mathcal{W}}_{2d}(0, m) = \frac{(m)_{2n-1}(2d-m-1)! \prod_{i=0}^{n-2}((2i+1)!(2d+2i)!)}{(d-m)! \prod_{i=0}^{2n-2}(d+i)!}. \tag{5.18}$$

Proof. Using (2.9) with $y = 0$ the recurrence relation (4.25) may be written as

$$\acute{R}^{(n)}_{2d}(0; \kappa) = \kappa^{-2}\frac{(d+2)_{d+1}}{(2n-2)_{d+1}}\left[\frac{(d+1)_{2n-2}}{(2d+1)_{2n-2}}\acute{R}^{(n-1)}_{2d+4}(0; \kappa) - \acute{R}^{(n-1)}_{2d+2}(0; \kappa)\right] \tag{5.19}$$

and equating coefficients

$$\acute{r}^{(n)}_{2d}(0; m) = \frac{(d+2)_{d+1}}{(2n-2)_{d+1}}\left[\frac{(d+1)_{2n-2}}{(2d+1)_{2n-2}}\acute{r}^{(n-1)}_{2d+4}(0; m+2) - \acute{r}^{(n-1)}_{2d+2}(0; m+2)\right]. \tag{5.20}$$

The theorem is true for $n = 1$ and for general n it follows by induction after extensive manipulation using (5.20). □

6. Conclusion

In this paper we have proved several theorems about non-intersecting lattice paths above a surface having a given number of contacts with the surface. The proofs are both analytical and combinatorial (using involutions and bijections).

We have found several new product forms for various special cases, in particular we show that the coefficients of the return and marked return polynomials for watermelons with fixed deviation can be expressed in terms of product forms.

Finally, we have derived two partial recurrence relations for particular cases of the n-path return and marked return polynomials.

Our results are restricted to watermelon configurations with fixed endpoint deviations. Extension to star configurations with fixed and free end point conditions may be possible in the future. Product forms for watermelons with free endpoints are unlikely to exist since they have not been found for the bulk and non-interacting surface cases. Other future work will be on the introduction of contact interactions between the chains.

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